

# On some exponential integral functionals of BM( $\mu$ ) and BES(3)

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## Abstract

In this paper we derive the Laplace transforms of the integral functionals

$$\int_0^\infty \left( p \left( \exp(B_t^{(\mu)}) + 1 \right)^{-1} + q \left( \exp(B_t^{(\mu)}) + 1 \right)^{-2} \right) dt,$$

and

$$\int_0^\infty \left( p \left( \exp(R_t^{(3)}) - 1 \right)^{-1} + q \left( \exp(R_t^{(3)}) - 1 \right)^{-2} \right) dt,$$

where  $p$  and  $q$  are real numbers,  $\{B_t^{(\mu)} : t \geq 0\}$  is a Brownian motion with drift  $\mu > 0$ , BM( $\mu$ ), and  $\{R_t^{(3)} : t \geq 0\}$  is a 3-dimensional Bessel process, BES(3). The transforms are given in terms of Gauss' hypergeometric functions and it is seen that the results are closely related to some functionals of Jacobi diffusions. This work generalizes and completes some results of Donati–Martin and Yor [4] and Salminen and Yor [12].

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processes.

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## 1 Introduction

Let  $B^{(\mu)} = \{B_t^{(\mu)} := B_t + \mu t : t \geq 0\}$  be a Brownian motion with drift  $\mu > 0$  started from 0. For the non-negative and locally integrable function  $f$  consider the functional

$$I_\infty(f) := \int_0^\infty f(B_s^{(\mu)}) ds.$$

It is known, see Engelbert and Senf [6] and Salminen and Yor [13],

$$I_\infty(f) < \infty \text{ a.s.} \Leftrightarrow \int_0^\infty f(x) dx < \infty. \quad (1.1)$$

In [13] also other properties of  $I_\infty(f)$  are studied; in particular, existence of moments.

During the recent years and because of applications in financial mathematics much interest has been focused on exponential functionals of  $B^{(\mu)}$ . We refer to Yor [18] for a collection of papers on this topic and also for further references. A particular rôle in these investigations has been played by the functional

$$\int_0^\infty \exp(-2B_s^{(\mu)}) ds \quad (1.2)$$

which, as shown in Dufresne [5], can be obtained as a perpetuity in a discrete pension funding scheme after a limiting procedure. See also Yor [17].

In [11] and [12] the functional

$$\int_0^\infty (a + \exp(B_s^{(\mu)}))^{-2} ds \quad (1.3)$$

is analyzed the motivation being that the functional in (1.3) has all the moments (in fact, some exponential moments) when Dufresne's functional (1.2) has only some moments. The distribution of the functional in (1.3) is computed in [11] for  $\mu = 1/2$  on one hand by connecting the functional to a hitting time of another Brownian motion with drift via random time change and on the other hand using the Feynman-Kac formula.

In this paper we derive the Laplace transform of the functional in (1.3) for arbitrary  $\mu > 0$ . In fact, we study the more general functional

$$\int_0^\infty \left( p \left( \exp(B_t^{(\mu)}) + 1 \right)^{-1} + q \left( \exp(B_t^{(\mu)}) + 1 \right)^{-2} \right) dt. \quad (1.4)$$

Notice that choosing here, e.g. ,  $p = -q = 1$  we obtain the functional

$$\int_0^\infty \cosh^{-2}(B_t^{(\mu)}/2) dt. \quad (1.5)$$

The key observation is that the functional in (1.4) can be expressed, roughly speaking, via the first hitting time of a Jacobi diffusion the Laplace transform of which can be computed explicitly in terms of Gauss' hypergeometric functions. It is also seen that similar techniques apply to compute the Laplace transform of an analogous functional for three-dimensional Bessel process  $\{R_t; t \geq 0\}$ :

$$\int_0^\infty \left( p \left( \exp(R_t - 1) \right)^{-1} + q \left( \exp(R_t) - 1 \right)^{-2} \right) dt. \quad (1.6)$$

The paper is organized so that in the next section some general results on the random time techniques connecting integral functionals and hitting times are presented. In this section we also discuss the Feynman–Kac approach. Definitions and some properties of Gauss' hypergeometric functions are given in Section 3. In Section 4 we discuss some properties of Jacobi diffusions focusing on two special cases emerging from our computations. The functional in (1.4) is treated in Section 5 and (1.6) in Section 6.

## 2 Transforming diffusions by changing time and space

In this section we give a fairly straightforward generalization, needed in Section 4, of the main result in Salminen and Yor [12] for quite general diffusions. We also discuss the corresponding Feynman–Kac approach.

Let  $B$  be a Brownian motion and consider a diffusion  $Y$  on an open interval  $I = (l, r)$  determined by the SDE

$$dY_t = \sigma(Y_t) dB_t + b(Y_t) dt. \quad (2.1)$$

It is assumed that  $\sigma$  and  $b$  are continuous and  $\sigma(x) > 0$  for  $x \in I$ . The diffusion  $Y$  is considered up to its explosion (or life) time

$$\zeta := \inf\{t : Y_t \notin I\}.$$

**Proposition 2.1.** *Let  $g(x)$ ,  $x \in I$ , be a twice continuously differentiable function such that  $g'(x) \neq 0$ . Consider the integral functional*

$$A_t := \int_0^t (g'(Y_s)\sigma(Y_s))^2 ds, \quad t \in [0, \zeta),$$

and its inverse

$$a_t := \min\{s : A_s > t\}, \quad t \in [0, A_\zeta).$$

Then the process  $Z$  given by

$$Z_t := g(Y_{a_t}), \quad t \in [0, A_\zeta), \quad (2.2)$$

is a diffusion satisfying the SDE

$$dZ_t = d\tilde{B}_t + G(g^{-1}(Z_t)) dt, \quad t \in [0, A_\zeta). \quad (2.3)$$

where  $\tilde{B}_t$  is a Brownian motion and

$$G(x) = (g'(x)\sigma(x))^{-2} \left( \frac{1}{2} \sigma(x)^2 g''(x) + b(x) g'(x) \right). \quad (2.4)$$

*Proof.* By Ito's formula for  $u \leq \zeta$

$$g(Y_u) - g(Y_0) = \int_0^u g'(Y_s)\sigma(Y_s) dB_s + \int_0^u (2^{-1}g''(Y_s)\sigma^2(Y_s) + g'(Y_s)b(Y_s)) ds.$$

Replacing  $u$  by  $a_t$  yields

$$Z_t - Z_0 = \int_0^{a_t} g'(Y_s)\sigma(Y_s) dB_s + \int_0^{a_t} (g'(Y_s)\sigma(Y_s))^2 G(Y_s) ds.$$

Since  $a_t$  is the inverse of  $A_t$  and  $A'_s = (g'(Y_s)\sigma(Y_s))^2$  we have

$$a'_t = \frac{1}{A'_{a_t}} = (g'(Y_{a_t})\sigma(Y_{a_t}))^{-2}. \quad (2.5)$$

From Lévy's theorem it follows that

$$\tilde{B}_t := \int_0^{a_t} g'(Y_s) \sigma(Y_s) dB_s, \quad t \in [0, A_\zeta).$$

is a (stopped) Brownian motion. As a result we obtain for  $t < A_\eta$

$$\begin{aligned} Z_t - Z_0 &= \tilde{B}_t + \int_0^t (g'(Y_{a_s}) \sigma(Y_{a_s}))^2 G(Y_{a_s}) da_s \\ &= \tilde{B}_t + \int_0^t G(g^{-1}(Z_s)) ds. \end{aligned}$$

□

It is interesting and useful to understand how the relation (2.2) expresses itself on the level of differential equations associated with the diffusions  $Y$  and  $Z$ . In particular, we consider in this framework the Feynman–Kac formula. First, we give the following proposition which is a generalization of the result for Brownian motion stated in [3] VI.15 p.113 and concerns distributions of functionals of Brownian motion stopped according to the inverse of an additive functional.

**Proposition 2.2.** *Let  $F$  and  $f$  be continuous functions on  $I$ . Assume also that  $F$  is bounded and  $f$  is non-negative. Let  $\tau$  be an exponentially (with parameter  $\lambda$ ) distributed random variable independent of  $Y$ . Then the function*

$$U(x) := \mathbf{E}_x \left( F(Y_{a_\tau}) \exp \left( - \int_0^{a_\tau} f(Y_s) ds \right) \right)$$

*is the unique bounded solution of the differential equation*

$$\begin{aligned} \frac{\sigma^2(x)}{2} u''(x) + b(x) u'(x) - (\lambda (g'(x) \sigma(x))^2 + f(x)) u(x) \\ = -\lambda (g'(x) \sigma(x))^2 F(x). \end{aligned} \quad (2.6)$$

Next, introduce for  $x \in J := g(I)$

$$Q(x) := U(g^{-1}(x)). \quad (2.7)$$

Straightforward calculations show that the function  $Q$  satisfies

$$\begin{aligned} \frac{1}{2} Q''(x) + G(g^{-1}(x)) Q'(x) - \left( \lambda + \frac{f(g^{-1}(x))}{(g'(g^{-1}(x)) \sigma(g^{-1}(x)))^2} \right) Q(x) \\ = -\lambda F(g^{-1}(x)). \end{aligned} \quad (2.8)$$

By the classical Feynman–Kac result the equation (2.8) has only one bounded and non-negative solution, and, consequently, we have the following probabilistic interpretation

$$Q(x) = \mathbf{E}_x \left( F(g^{-1}(Z_\tau^\circ)) \exp \left( - \int_0^\tau \frac{f(g^{-1}(Z_s^\circ))}{(g'(g^{-1}(Z_s^\circ))\sigma(g^{-1}(Z_s^\circ)))^2} ds \right) \right),$$

where the diffusion  $Z^\circ$  has the generator

$$\frac{1}{2} \frac{d^2}{dx^2} + G(g^{-1}(x)) \frac{d}{dx},$$

and can be viewed as a solution of the SDE (2.3). Using (2.7) and the probabilistic interpretations of  $U$  and  $Q$  we obtain

$$\begin{aligned} & \mathbf{E}_{g^{-1}(x)} \left( F(Y_{a_\tau}) \exp \left( - \int_0^\tau \frac{f(Y_{a_s})}{(g'(Y_{a_s})\sigma(Y_{a_s}))^2} ds \right) \right) \\ &= \mathbf{E}_x \left( F(g^{-1}(Z_\tau^\circ)) \exp \left( - \int_0^\tau \frac{f(g^{-1}(Z_s^\circ))}{(g'(g^{-1}(Z_s^\circ))\sigma(g^{-1}(Z_s^\circ)))^2} ds \right) \right). \end{aligned} \quad (2.9)$$

From this identity we can recover, e.g., that the distributions of  $Y_{a_t}$  and  $Z_t^\circ$  are equal (from (2.2) we, of course, know much more). In this paper, see Sections 4.2 and 4.3, the identity (2.9) is used in a particular case to find the solutions of the equation (2.8) in terms of the (known) solutions of the equation (2.6). Notice also that Proposition 2.2 can be proved by using first the classical Feynman–Kac formula for (2.8) and then Proposition 2.1.

Next two propositions connect occupation time functionals with first hitting times. We use the notation in Proposition 2.1.

**Proposition 2.3. 1.** *Let  $x \in I$  and  $y \in I$  be such that  $P_x$ -a.s.*

$$H_y(Y) := \inf\{t : Y_t = y\} < \infty.$$

*Then, under the assumptions in Proposition 2.1,*

$$A_{H_y(Y)} = \inf\{t : Z_t = g(y)\} =: H_{g(y)}(Z) \quad a.s. \quad (2.10)$$

*with  $Y_0 = x$  and  $Z_0 = g(x)$ .*

**2.** Assume  $g(r) := \lim_{z \rightarrow r} g(z)$  exists (recall that  $r$  denotes the right hand side end point of  $I$ ). Suppose also that each of the following statements holds a.s.

$$(i) \quad \zeta = +\infty, \quad (ii) \quad \lim_{t \rightarrow \infty} Y_t = r, \quad (iii) \quad A_\infty := \lim_{t \rightarrow +\infty} A_t < \infty. \quad (2.11)$$

Then

$$A_\infty = H_{g(r)}(Z) \quad a.s. \quad (2.12)$$

*Proof.* To prove (2.10), notice that from the definition of  $Z$  (see (2.2)) we obtain

$$Z_{A_t} = g(Y_t).$$

Letting here  $t \rightarrow H_y(Y)$  and using the fact that  $g$  is monotone yield (2.10). The claim (2.12) is proved similarly.  $\square$

**Remark 2.4.** A sufficient condition for (iii) in (2.11) is clearly that the mean of  $A_\infty$  is finite:

$$\begin{aligned} \mathbf{E}_x(A_\infty) &= \int_0^\infty \mathbf{E}_x \left( (g'(Y_s) \sigma(Y_s))^2 \right) ds \\ &= \int_l^r G_0(x, y) (g'(y) \sigma(y))^2 m(dy) < \infty, \end{aligned}$$

where  $G_0$  denotes the Green kernel of  $Y$  and  $m$  is the speed measure (for these see, e.g., Borodin and Salminen [3]). For a necessary and sufficient condition in the case of a Brownian motion with drift, see (1.1).

For the next proposition recall from (2.5) and (2.2) that the inverse  $a_t$  of  $A_t$  is given for  $t < A_\zeta$  by

$$a_t = \int_0^t (g'(g^{-1}(Z_s)) \sigma(g^{-1}(Z_s)))^{-2} ds. \quad (2.13)$$

**Proposition 2.5.** Assume that each of the following statements holds a.s.

$$(i) \quad \zeta < \infty, \quad (ii) \quad \lim_{t \rightarrow \zeta} Y_t = l, \quad (iii) \quad A_\zeta = \infty.$$

Then

$$a_\infty := \int_0^\infty (g'(g^{-1}(Z_s)) \sigma(g^{-1}(Z_s)))^{-2} ds = \zeta \quad a.s. \quad (2.14)$$

*Proof.* The claim is immediate from (2.13) using the assumptions (i)–(iii) and the fact that  $a_t$  is the inverse of  $A_t$ . Notice also that in this case  $\zeta = \lim_{x \rightarrow l} H_x(Y)$ .  $\square$

We conclude this section by a proposition which relates the results in Propositions 2.3 and 2.5 to the Feynman–Kac formula. We consider only the case in Proposition 2.3.2 and leave the other cases to the reader.

**Proposition 2.6. 1.** *Under the assumptions of Proposition 2.3.2, the function*

$$\Phi(x) := \mathbf{E}_x \left( \exp \left( -\rho \int_0^\infty (g'(Y_s)\sigma(Y_s))^2 ds \right) \right), \quad x \in I, \quad \rho > 0$$

*is the unique increasing positive solution of the differential equation*

$$\frac{1}{2} \sigma^2(x) \Phi''(x) + b(x) \Phi'(x) - \rho (g'(x)\sigma(x))^2 \Phi(x) = 0, \quad (2.15)$$

*satisfying  $\lim_{x \rightarrow r} \Phi(x) = 1$ .*

**2.** *Define for  $x \in g^{-1}(I)$*

$$\Psi(x) := \Phi(g^{-1}(x)).$$

*Then  $\Psi$  is the unique increasing positive solution of the differential equation*

$$\frac{1}{2} \Psi''(x) + G(g^{-1}(x)) \Psi'(x) - \rho \Psi(x) = 0$$

*satisfying  $\lim_{x \rightarrow r} \Psi(x) = 1$ . Moreover,*

$$\begin{aligned} \Psi(x) &= \mathbf{E}_x \left( \exp(-\rho H_{g(r)}(Z)) \right) \\ &= \mathbf{E}_{g^{-1}(x)} \left( \exp \left( -\rho \int_0^\infty (g'(Y_s)\sigma(Y_s))^2 ds \right) \right). \end{aligned} \quad (2.16)$$

*Proof.* The first part of the proposition can be proved as Proposition XX in Salminen and Yor [12]. The fact that  $\Psi$  has the claimed properties is an easy exercise in differentiation. The first equality in (2.16) is standard diffusion theory and the second one follows from the definition of  $\Psi$ .  $\square$



### 3 Gauss' hypergeometric functions

We start with by recalling from Abramowitz and Stegun [1] p. 556, the definition of Gauss' hypergeometric series (or functions):

$$F(\alpha, \beta, \gamma; x) := \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} \frac{x^n}{n!}. \quad (3.1)$$

It is easily seen that  $F$  is well defined for  $|x| < 1$  but we consider  $F$  only for real values on  $x$  such that  $0 < x < 1$ . However, we allow complex conjugate values for  $\alpha$ , and  $\beta$ , but take  $\gamma$  real (see Theorems 5.1 and 6.1). Notice that

$$F(\alpha, \beta, \gamma; x) = F(\beta, \alpha, \gamma; x).$$

It is well known (see Abramowitz and Stegun [1] or Lebedev [9] p. 162) that  $F$  is a solution of the ODE

$$x(1-x)v''(x) + (\gamma - (\alpha + \beta + 1)x)v'(x) - \alpha\beta v(x) = 0. \quad (3.2)$$

Straightforward calculations show that also

$$\widehat{F}(\alpha, \beta, \gamma; x) := F(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - x) \quad (3.3)$$

is a solution of (3.2).

Recall also that for  $0 < x < 1$ , and  $\text{Re}(\gamma) > \text{Re}(\alpha) > 0$  it holds

$$F(\alpha, \beta, \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha-1} (1-t)^{\gamma-\alpha-1} (1-tx)^{-\beta} dt. \quad (3.4)$$

From (3.4) we obtain by changing variables

$$\begin{aligned} \widehat{F}(\alpha, \beta, \gamma; x) &= F(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - x) \\ &= \frac{\Gamma(\alpha + \beta + 1 - \gamma)}{\Gamma(\alpha)\Gamma(\beta + 1 - \gamma)} \int_0^{\infty} t^{\alpha-1} (1+t)^{\gamma-\alpha-1} (1+tx)^{-\beta} dt, \end{aligned} \quad (3.5)$$

which is valid for  $0 < x < 1$ ,  $\text{Re}(\beta + 1 - \gamma) > 0$ , and  $\text{Re}(\alpha) > 0$ . Notice that

$$\widehat{F}(\alpha, \beta, \gamma; 1) = 1$$

and

$$\widehat{F}(\alpha, \beta, \gamma; 0) = \frac{\Gamma(\alpha + \beta + 1 - \gamma) \Gamma(1 - \gamma)}{\Gamma(\alpha + 1 - \gamma) \Gamma(\beta + 1 - \gamma)}, \quad (3.6)$$

where all the arguments in our case (see Theorem 5.1) are positive or have positive real parts.

## 4 Jacobi diffusion

### 4.1 Definition and two particular cases

Consider the diffusion  $X = \{X_t : t \geq 0\}$  living in  $(0, 1)$  satisfying the SDE

$$dX_t = \sqrt{2X_t(1 - X_t)} dB_t + (\gamma - (\alpha + \beta + 1)X_t) dt. \quad (4.1)$$

Following Karlin and Taylor [8] p. 335,  $X$  is called a Jacobi diffusion with parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ . See the papers by Mazet [10], Warren and Yor [16], Hu, Shi and Yor [7], Schoutens [14], and Albanese and Kuznetsov [2] for some results and applications of Jacobi diffusions (and also for further references). Next proposition gives a basic simple property of Jacobi diffusions.

**Proposition 4.1.** *Let  $X$  be a Jacobi diffusion with parameters  $\alpha$ ,  $\beta$ , and  $\gamma$ . Then  $Y := 1 - X$  is a Jacobi diffusion with parameters  $\alpha$ ,  $\beta$ , and  $\alpha + \beta + 1 - \gamma$ .*

The generator of  $X$  is given by

$$\mathcal{G}u(x) = x(1 - x)u''(x) + (\gamma - (\alpha + \beta + 1)x)u'(x),$$

and we use

$$S(x) = \int_{1/2}^x y^{-\gamma}(1 - y)^{\gamma - \alpha - \beta - 1} dy$$

as the scale, and

$$m(x_1, x_2) = \int_{x_1}^{x_2} y^{\gamma-1}(1 - y)^{\alpha+\beta-\gamma} dy, \quad 0 \leq x_1 < x_2 \leq 1$$

as the speed. Obviously,  $X$  exhibits very different behaviour when the values of the parameters  $\alpha$ ,  $\beta$ , and  $\gamma$  are varied. In view of our applications we consider two cases.

**Case 1:**  $\gamma < 1$  and  $\alpha + \beta > 1$ .

Here,  $S(0+) > -\infty$  and  $S(1-) = +\infty$ , and by the standard boundary point analysis (see, e.g., Borodin and Salminen [3])

- if  $\gamma \leq 0$  then 0 is exit-entrance (regular),
- if  $0 < \gamma < 1$  then 0 is exit-not-entrance,

- 1 is entrance-not-exit.

Consequently,  $X$  is transient and

$$H_0(X) := \inf\{t : X_t = 0\} < \infty \quad \text{a.s.}$$

In the regular case, i.e.,  $\gamma \leq 0$ , we choose 0 to be a killing boundary. We remark that the boundaries 0 and 1 are as displayed above also if the condition  $\alpha + \beta > 1$  is extended to  $\alpha + \beta > \gamma - 1$ .

**Case 2:**  $\gamma = \alpha + \beta \geq 2$ .

Now we have  $S(0+) = -\infty$  and  $S(1-) = +\infty$ ,  $m(0, 1) < \infty$ , and

- 0 is entrance-not-exit,
- 1 is entrance-not-exit.

Hence,  $X$  is positively recurrent. Notice, cf. Proposition 4.1, that if  $X^{(1)}$  satisfies (4.1) with  $\gamma = \alpha + \beta$  then  $X^{(2)} := 1 - X^{(1)}$  solves (4.1) with  $\gamma = 1$ . This fact is also transparent in the expressions for  $S$  and  $m$ .

## 4.2 Functionals of Jacobi diffusion; Case 1

Consider the Jacobi diffusion given by (4.1) with  $\gamma < 1$  and  $\alpha + \beta > 1$ . Let for  $c, \theta \geq 0$

$$f(x) := \frac{1}{\theta} \log \left( \frac{1-x}{cx} \right), \quad 0 < x < 1.$$

Clearly,  $f$  is decreasing,  $f(0+) = +\infty$ ,  $f(1-) = -\infty$ , and

$$f^{-1}(x) = \frac{1}{c e^{\theta x} + 1}.$$

We are interested in the process  $\{f(X_t) : t \geq 0\}$ . By Ito's formula for  $t < H_0(X)$

$$\begin{aligned} & \log \left( \frac{1-X_t}{c X_t} \right) - \log \left( \frac{1-X_0}{c X_0} \right) \\ &= \int_0^t \frac{-\sqrt{2}}{\sqrt{X_s(1-X_s)}} dB_s + \int_0^t \frac{1-\gamma + (\alpha + \beta - 1)X_s}{X_s(1-X_s)} ds. \end{aligned} \quad (4.2)$$

Define for  $t < H_0(X)$

$$A_t := \frac{2}{\theta^2} \int_0^t \frac{1}{X_s(1 - X_s)} ds,$$

and set  $A_t = +\infty$  for  $t \geq H_0(X)$ . Let  $a$  be the inverse of  $A$ , i.e.,

$$a_t := \inf\{s : A_s > t\},$$

and notice that  $a_t \leq H_0(X)$  for all  $t \geq 0$ . Moreover,

$$a_t = \frac{\theta^2}{2} \int_0^t X_{a_s}(1 - X_{a_s}) ds. \quad (4.3)$$

The process given by

$$Z_t := \frac{1}{\theta} \log \left( \frac{1 - X_{a_t}}{c X_{a_t}} \right) \quad (4.4)$$

is well defined for all  $t$  such that  $a_t < H_0(X)$ , and from (4.2) we obtain

$$Z_t - Z_0 = B_t^\circ + \frac{\theta}{2} \int_0^t \left( 1 - \gamma + \frac{\alpha + \beta - 1}{c e^{\theta Z_s} + 1} \right) ds, \quad (4.5)$$

where  $B^\circ$  is a Brownian motion. From (4.5) it is seen that  $Z$  is, in fact, non-exploding and, therefore, from (4.4) it follows that  $a_t < H_0(X)$  for all  $t$  and  $a_t \rightarrow H_0(X)$  as  $t \rightarrow \infty$ . Hence, it also holds that  $A_t \rightarrow \infty$  as  $t \rightarrow H_0(X)$ , and we are in the case treated in Proposition 2.5. We have

$$\begin{aligned} \int_0^t \frac{c \exp(\theta Z_s)}{(c \exp(\theta Z_s) + 1)^2} ds &= \int_0^t \frac{1 - X_{a_s}}{X_{a_s}} \left( \frac{1 - X_{a_s}}{X_{a_s}} + 1 \right)^{-2} ds \\ &= \int_0^t X_{a_s}(1 - X_{a_s}) ds \\ &= \frac{2}{\theta^2} a_t, \end{aligned}$$

and the statement in Proposition 2.5 can be formulated as follows

**Proposition 4.2.** *Let  $Z$ ,  $Z_0 = x$ , and  $X$ ,  $X_0 = 1/(c e^{\theta x} + 1)$ , be as above. Then*

$$\frac{\theta^2}{2} \int_0^\infty \frac{c \exp(\theta Z_s)}{(c \exp(\theta Z_s) + 1)^2} ds = H_0(X) \quad \text{a.s.} \quad (4.6)$$

As explained in Section 3, we can use the Feynman–Kac method to deduce that the Laplace-transforms of the functionals in (4.6) are equal. However, because the present case is not covered by Proposition 2.6, we formulate here a result connecting the solutions of the hypergeometric differential equation to the solutions of the equation induced by the generator of  $Z$  with the potential term as on the left hand side of (4.6).

**Proposition 4.3.** *Let  $z(x) = z(\alpha, \beta, \gamma, x)$ ,  $x \in (0, 1)$ , be an arbitrary solution of the hypergeometric differential equation*

$$x(1-x)z''(x) + (\gamma - (\alpha + \beta + 1)x)z'(x) - \alpha\beta z(x) = 0. \quad (4.7)$$

Then for  $c \geq 0$  and  $\theta \in \mathbf{R}$  the function  $q(x) = z(y(x))$  with

$$y(x) := (ce^{\theta x} + 1)^{-1}$$

satisfies for  $x \in (-\infty, \infty)$  the equation

$$q''(x) + \theta \left(1 - \gamma + \frac{\alpha + \beta - 1}{ce^{\theta x} + 1}\right) q'(x) - \frac{\theta^2 \alpha \beta c e^{\theta x}}{(ce^{\theta x} + 1)^2} q(x) = 0. \quad (4.8)$$

*Proof.* It is sufficient to prove this statement for  $a = 1$  since one can shift the argument. Differentiating the composition of the functions we have (the corresponding arguments are omitted)

$$q' = -\frac{\theta e^{\theta x}}{(e^{\theta x} + 1)^2} z', \quad q'' = \frac{\theta^2 e^{2\theta x}}{(e^{\theta x} + 1)^4} z'' + \frac{\theta^2 e^{\theta x} (e^{\theta x} - 1)}{(e^{\theta x} + 1)^3} z'.$$

Taking in (4.7) the argument  $y(x)$  instead of  $x$ , one obtains

$$\frac{e^{\theta x}}{(e^{\theta x} + 1)^2} z'' + \left(\gamma - \frac{\alpha + \beta + 1}{e^{\theta x} + 1}\right) z' - \alpha\beta z = 0.$$

Finally, from these relations we have

$$\begin{aligned} q''(x) &= \frac{\theta^2 e^{\theta x}}{(e^{\theta x} + 1)^2} \left( \left( \frac{\alpha + \beta + 1}{e^{\theta x} + 1} - \gamma + \frac{e^{\theta x} - 1}{e^{\theta x} + 1} \right) z' + \alpha\beta z \right) \\ &= -\theta \left( \frac{e^{\theta x} + \alpha + \beta}{e^{\theta x} + 1} - \gamma \right) q'(x) + \frac{\theta^2 \alpha \beta e^{\theta x}}{(e^{\theta x} + 1)^2} q(x) \end{aligned}$$

proving the equation (4.8). □

**Remark 4.4.** Notice also the converse of Proposition 4.3: if  $q$  is a solution of (4.8) then  $z(x) = q(y^{-1}(x))$ , where

$$y^{-1}(x) := \frac{1}{\theta} \log \left( \frac{1-x}{cx} \right), \quad 0 < x < 1,$$

is a solution of (4.7) (cf. (4.4)).

### 4.3 Functionals of Jacobi diffusion; Case 2

Assume that  $X$  is a Jacobi diffusion with  $\gamma = \alpha + \beta \geq 2$  and recall that in this case  $X$  is recurrent. By Ito's formula

$$\begin{aligned} & \log(1 - X_t) - \log(1 - X_0) \\ &= -\sqrt{2} \int_0^t \sqrt{\frac{X_s}{1 - X_s}} dB_s \\ & \quad - \int_0^t \frac{\gamma - (\alpha + \beta + 1)X_s}{1 - X_s} ds - \frac{1}{2} \int_0^t \frac{2X_s(1 - X_s)}{(1 - X_s)^2} ds \\ &= -\sqrt{2} \int_0^t \sqrt{\frac{X_s}{1 - X_s}} dB_s - \int_0^t \frac{X_s}{1 - X_s} \left( \frac{\gamma}{X_s} - \alpha - \beta \right) ds \quad (4.9) \\ &= -\sqrt{2} \int_0^t \sqrt{\frac{X_s}{1 - X_s}} dB_s - \int_0^t \frac{X_s}{1 - X_s} \frac{(\alpha + \beta)(1 - X_s)}{X_s} ds. \end{aligned}$$

Let

$$\hat{A}_t := \frac{2}{\theta^2} \int_0^t \frac{X_s}{1 - X_s} ds,$$

and define its inverse  $\hat{a}_t := \inf\{s : \hat{A}_s > t\}$ . By continuity,  $\hat{A}_t < \infty$  and  $\hat{a}_t < \infty$  for all  $t$  and  $\hat{A}_t \rightarrow \infty$  and  $\hat{a}_t \rightarrow \infty$  as  $t \rightarrow \infty$ ; hence, we are in the case covered by Proposition 2.3.1. Instead of simply referring to Proposition 2.3.1, we give some details. Firstly

$$\hat{a}_t = \frac{\theta^2}{2} \int_0^t \frac{1 - X_{a_s}}{X_{a_s}} ds, \quad (4.10)$$

and the process given by

$$\hat{Z}_t := -\frac{1}{\theta} \log(1 - X_{\hat{a}_t}) \quad (4.11)$$

is well defined for all  $t \geq 0$ . From (4.9) we obtain

$$\widehat{Z}_t - \widehat{Z}_0 = \widehat{B}_t^\circ + \frac{\theta}{2} \int_0^t \frac{\alpha + \beta}{\exp(\theta \widehat{Z}_s) - 1} ds$$

where  $\widehat{B}^\circ$  is a Brownian motion. From (4.10) and (4.11) we obtain

$$\widehat{a}_t = \frac{\theta^2}{2} \int_0^t \frac{1 - X_{a_s}}{X_{a_s}} ds = \frac{\theta^2}{2} \int_0^t \left( \exp(\theta \widehat{Z}_s) - 1 \right)^{-1} ds$$

For  $y > 0$  introduce

$$H_y(\widehat{Z}) := \inf\{t : \widehat{Z}_t = y\}.$$

Now, from (4.11),

$$y = \widehat{Z}_{H_y(\widehat{Z})} = -\frac{1}{\theta} \log \left( 1 - X_{\widehat{a}_{H_y(\widehat{Z})}} \right)$$

and, hence,

$$\widehat{a}_{H_y(\widehat{Z})} = \inf\{t : X_t = 1 - e^{-\theta y}\}.$$

Consequently, we arrive to the result (cf. Proposition 2.3.1)

**Proposition 4.5.** *Let  $X$  be a Jacobi diffusion with  $\gamma = \alpha + \beta \geq 2$  and  $\widehat{Z}$  as defined in (4.11) with  $\widehat{Z}_0 = x$ . Then for  $y > 0$*

$$\frac{\theta^2}{2} \int_0^{H_y(\widehat{Z})} \left( \exp(\theta \widehat{Z}_s) - 1 \right)^{-1} ds = H_{y^*}(X). \quad (4.12)$$

where  $y^* = 1 - e^{-\theta y}$  and  $X_0 = 1 - e^{-\theta x}$ .

It is useful (again) to give the corresponding result for differential equations. Notice that in this result we do not interpret solutions probabilistically and, therefore, can formulate a more general statement without any restrictions on the values of the parameters.

**Proposition 4.6.** *Let  $z(x) = z(\alpha, \beta, \gamma, x)$ ,  $x \in (0, 1)$ , be an arbitrary solution of the hypergeometric differential equation (4.7). Then for  $\theta > 0$  the function  $s(x) = z(1 - e^{-\theta x})$  satisfies for  $x \in (0, \infty)$  the equation*

$$s''(x) + \theta \left( \frac{\gamma e^{\theta x}}{e^{\theta x} - 1} - \alpha - \beta \right) s'(x) - \frac{\theta^2 \alpha \beta}{e^{\theta x} - 1} s(x) = 0.$$

*Proof.* The proof is analogous to the proof of Proposition 4.3. We have

$$s' = \theta e^{-\theta x} z', \quad s'' = \theta^2 e^{-2\theta x} z'' - \theta^2 e^{-\theta x} z'.$$

Taking in (3.2) the argument  $1 - e^{-\theta x}$  instead of  $x$  yields

$$e^{-\theta x} z'' - \left( \alpha + \beta + 1 - \frac{\gamma}{1 - e^{-\theta x}} \right) z' - \frac{\alpha\beta}{1 - e^{-\theta x}} z = 0.$$

Consequently,

$$\begin{aligned} s''(x) &= \theta^2 e^{-\theta x} \left( \alpha + \beta + 1 - \frac{\gamma}{1 - e^{-\theta x}} - 1 \right) z' + \frac{\theta^2 \alpha \beta e^{-\theta x}}{1 - e^{-\theta x}} z \\ &= \theta \left( \alpha + \beta - \frac{\gamma}{1 - e^{-\theta x}} \right) s'(x) + \frac{\theta^2 \alpha \beta e^{-\theta x}}{1 - e^{-\theta x}} s(x), \end{aligned}$$

proving the claim.  $\square$

**Remark 4.7. 1.** We have also

$$\begin{aligned} &\log(X_t) - \log(X_0) \\ &= \sqrt{2} \int_0^t \sqrt{\frac{1 - X_s}{X_s}} dB_s + \int_0^t \frac{1 - X_s}{X_s} \left( \alpha + \beta - \frac{\alpha + \beta + 1 - \gamma}{1 - X_s} \right) ds. \end{aligned}$$

Choosing  $\gamma = 1$  and letting

$$A_t^\circ := \frac{2}{\theta^2} \int_0^t \frac{1 - X_s}{X_s} ds,$$

it is seen that the process

$$Z_t^\circ := -\frac{1}{\theta} \log(X_{a_t^\circ}),$$

where  $a^\circ$  is the inverse of  $A^\circ$ , satisfies the same SDE as  $\widehat{Z}$ .

**2.** Let  $z(x) = z(\alpha, \beta, \gamma, x)$ ,  $x \in (0, 1)$ , be an arbitrary solution of the hypergeometric differential equation (4.7). Then for  $\theta \geq 0$  the function  $q(x) = z(e^{-\theta x})$  satisfies for  $x > 0$  the equation

$$q''(x) + \theta \left( 1 - \gamma + \frac{\alpha + \beta + 1 - \gamma}{e^{\theta x} - 1} \right) q'(x) - \frac{\theta^2 \alpha \beta}{e^{\theta x} - 1} q(x) = 0. \quad (4.13)$$



## 5 Perpetual integral functional of BM( $\mu$ )

Our first main result completes, in a sense, the result in [11], see also [12], concerning the translated Dufresne's functional but, moreover, it gives Laplace transforms for many new perpetual integral functionals. The functional we analyze is

$$I_\infty(p, q) := \int_0^\infty \left( \frac{p}{c \exp(\theta B_s^{(\mu)}) + 1} + \frac{q}{(c \exp(\theta B_s^{(\mu)}) + 1)^2} \right) ds.$$

The notation  $I_t(p, q)$  is used when the integration is from 0 to  $t$  and  $I_{H_y}(p, q)$  when  $t$  equals

$$H_y := \inf\{s : B_s^{(\mu)} = y\},$$

the first hitting time of  $y$ .

**Theorem 5.1.** *Let  $B^{(\mu)}$  be a Brownian motion with drift  $\mu > 0$  started from  $x$ . Then for  $c > 0$ ,  $\theta > 0$ ,  $p \geq 0$  and  $p + q \geq 0$*

$$\begin{aligned} \mathbf{E}_x \left( \exp \left( -I_\infty(p, q) \right) \right) \\ = K v(x)^k F(\alpha, \beta, \alpha + \beta + 2\mu/\theta; v(x)) \end{aligned} \quad (5.1)$$

where  $F$  is Gauss' hypergeometric function as defined in (3.1), and

$$k = (\alpha + \beta - 1)/2, \quad v(x) = \frac{c \exp(\theta x)}{c \exp(\theta x) + 1}$$

$$K = \frac{\Gamma(\alpha + 2\mu/\theta) \Gamma(\beta + 2\mu/\theta)}{\Gamma(\alpha + \beta + 2\mu/\theta) \Gamma(2\mu/\theta)} \quad (5.2)$$

and

$$\alpha = \frac{1}{2} - \mu/\theta + \sqrt{\mu^2 + 2(p+q)}/\theta + \frac{1}{2} \sqrt{1 + 8q/\theta^2}, \quad (5.3)$$

$$\beta = \frac{1}{2} - \mu/\theta + \sqrt{\mu^2 + 2(p+q)}/\theta - \frac{1}{2} \sqrt{1 + 8q/\theta^2}. \quad (5.4)$$

*Proof.* Recall, e.g., from [12] that

$$\begin{aligned} \Psi(x) &:= \mathbf{E}_x \left( \exp \left( -I_\infty(p, q) \right) \right) \\ &= \mathbf{E}_x \left( \exp \left( - \int_0^\infty \left( p \left( 1 - v(B_t^{(\mu)}) \right) + q \left( 1 - v(B_t^{(\mu)}) \right)^2 \right) dt \right) \right) \end{aligned}$$

is the unique positive bounded function such that

$$\frac{1}{2}\Psi''(x) + \mu\Psi'(x) - \left(p(1-v(x)) + q(1-v(x))^2\right)\Psi(x) = 0 \quad (5.5)$$

and  $\lim_{x \rightarrow +\infty} \Psi(x) = 1$ . We remark that in [12] it is required  $\Psi$  to be increasing, but from the proof therein it is clear that an equivalent requirement is boundedness. To find  $\Psi$  we use Girsanov's theorem and Proposition 4.2. Firstly, recall that the process  $Z$  as defined in (4.4) solves the SDE

$$dZ_t = dB_t^\circ + \frac{\theta}{2} \left(1 - \gamma + (\alpha + \beta - 1)(1 - v(Z_t))\right) dt.$$

Choose now  $\gamma$  such that  $\mu = \theta(1-\gamma)/2$ . By Girsanov's theorem, the measures  $\mathbf{P}^Z$  and  $\mathbf{P}^{(\mu)}$  induced by  $Z$  and  $B^{(\mu)}$ , respectively, and defined in the space of continuous functions  $\omega : \mathbf{R}_+ \mapsto \mathbf{R}$  are locally absolutely continuous with the Radon–Nikodym derivative given by

$$\mathbf{P}^Z|_{\mathcal{F}_t} = \exp(D_t) \mathbf{P}^{(\mu)}|_{\mathcal{F}_t}, \quad (5.6)$$

where  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by the coordinate process up to time  $t$  and

$$D_t := \int_0^t f(\omega_s) d\omega_s - \frac{1}{2} \int_0^t \left(f^2(\omega_s) + 2\mu f(\omega_s)\right) ds \quad (5.7)$$

with

$$f(\omega_s) = \theta k(1 - v(\omega_s)).$$

Observe

$$V(x) := \int^x (1 - v(y)) dy = \frac{1}{\theta} \int^x \frac{v'(y)}{v(y)} dy = \frac{1}{\theta} \log v(x).$$

For the first integral in (5.7) we obtain by Ito's formula

$$\begin{aligned} \int_0^t (1 - v(\omega_s)) d\omega_s &= V(\omega_t) - V(\omega_0) + \frac{1}{2} \int_0^t v'(\omega_s) ds \\ &= V(\omega_t) - V(\omega_0) + \frac{\theta}{2} \int_0^t v(\omega_s)(1 - v(\omega_s)) ds \\ &= V(\omega_t) - V(\omega_0) - \frac{\theta}{2} \int_0^t (1 - v(\omega_s))^2 ds \\ &\quad + \frac{\theta}{2} \int_0^t (1 - v(\omega_s)) ds. \end{aligned}$$

Letting (cf. Proposition 4.2)

$$I_t := \frac{\theta^2}{2} \int_0^t \frac{c \exp(\theta \omega_s)}{(c \exp(\theta \omega_s) + 1)^2} ds$$

we have

$$\begin{aligned} I_t &= \frac{\theta^2}{2} \int_0^t v(\omega_s)(1 - v(\omega_s)) ds \\ &= \frac{\theta^2}{2} \int_0^t (1 - v(\omega_s)) ds - \frac{\theta^2}{2} \int_0^t (1 - v(\omega_s))^2 ds. \end{aligned}$$

The absolute continuity relation (5.6) yields

$$\mathbf{E}_x^Z \left( \exp(-r I_t) \right) = \mathbf{E}_x^{(\mu)} \left( \exp(D_t - r I_t) \right). \quad (5.8)$$

Straightforward but lengthy computations show that choosing  $r = \alpha\beta$  the claimed expressions (5.3) and (5.4) for  $\alpha$  and  $\beta$ , respectively, are such that

$$\begin{aligned} &\mathbf{E}_x^{(\mu)} \left( \exp(D_t - r I_t) \right) \\ &= \mathbf{E}_x \left[ \exp \left( \theta k V(B_t^{(\mu)}) - \theta k V(x) \right) \exp \left( -I_t(p, q) \right) \right]. \end{aligned} \quad (5.9)$$

We remark that  $r = \alpha\beta$  can attain both positive and negative values. The equalities (5.8) and (5.9) hold also for the first hitting time  $H_y$  with  $y > x$ , i.e.,

$$\mathbf{E}_x^Z \left( \exp(-r I_{H_y}) \right) = \left( \frac{v(y)}{v(x)} \right)^k \mathbf{E}_x^{(\mu)} \left[ \exp \left( -I_{H_y}(p, q) \right) \right].$$

Letting  $y \rightarrow +\infty$  (remember that  $Z_t \rightarrow +\infty$  and  $B_t^{(\mu)} \rightarrow +\infty$  as  $t \rightarrow +\infty$ ) we obtain by monotone convergence and Proposition 4.2

$$\Psi(x) = (v(x))^k \mathbf{E}_{x^*}^X \left[ \exp \left( -\alpha \beta H_0(X) \right) \right], \quad (5.10)$$

where  $x^* := 1 - v(x)$ . Recall that  $\psi(x) \rightarrow 1$  as  $x \rightarrow +\infty$ . This can also be verified from (5.10) by observing that as  $x \rightarrow +\infty$  then  $x^* \rightarrow 0$  and the right hand side of (5.10) tends to 1 because 0 is an exit (or killing) boundary point

for  $X$ . Notice also that if  $p + q = 0$  (see Example 5.3) (in this case  $\alpha\beta > 0$ ) then  $k = 0$  and we have

$$0 < \lim_{x \rightarrow -\infty} \Psi(x) = \lim_{x^* \rightarrow 1} \mathbf{E}_{x^*}^X \left[ \exp \left( -\alpha\beta H_0(X) \right) \right].$$

Also if  $p + q > 0$  then  $\Psi(x) \rightarrow 0$  as  $x \rightarrow -\infty$ . In this case, because  $v(x) \rightarrow 0$  we cannot without further analysis claim that

$$\lim_{x^* \rightarrow 1} \mathbf{E}_{x^*}^X \left[ \exp \left( -\alpha\beta H_0(X) \right) \right] < +\infty$$

in case  $\alpha\beta < 0$  (but this will follow from our discussion !). For the general theory of diffusions we know that the function

$$\psi_{\alpha\beta}^X(x^*) := \mathbf{E}_{x^*}^X \left[ \exp \left( -\alpha\beta H_0(X) \right) \right]$$

is a solution of the hypergeometric differential equation

(Clearly,  $\psi_{\alpha\beta}^X$  as a function of  $x^*$  is decreasing if  $\alpha\beta > 0$ , and increasing if  $\alpha\beta < 0$ . Hence, because  $1 - v(x)$  is decreasing,  $\psi_{\alpha\beta}^X$  as a function of  $x$  is increasing if  $\alpha\beta > 0$ , and decreasing if  $\alpha\beta < 0$ . In the latter case, multiplying  $\psi_{\alpha\beta}^X$  with the increasing function  $v(x)^k$  makes the product increasing.)

Let  $F$  be the hypergeometric function as defined in (3.1) with  $\alpha$  and  $\beta$  as in (5.3) and (5.4), respectively, and  $\gamma = 1 - (2\mu/\theta)$ . Notice that  $\alpha$  and  $\beta$  can be conjugate complex numbers and

$$\alpha + \beta > 1, \quad \text{and} \quad \gamma < 1.$$

Moreover, as is easily seen,

$$\operatorname{Re}(\beta + 1 - \gamma) = \operatorname{Re}(\beta + (2\mu/\theta)) > 0.$$

From Section 3 it now follows (cf. (3.5)) that the function

$$x \mapsto F(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - x) = F(\alpha, \beta, \alpha + \beta + (2\mu/\theta); 1 - x)$$

is a bounded non-negative solution of the hypergeometric differential equation. Proposition 4.3 and formula (5.10) yield

$$\Psi^\circ(x) := (v(x))^k F(\alpha, \beta, \alpha + \beta + (2\mu/\theta); v(x))$$

is a nonnegative bounded solution of (5.5). Consequently, by the uniqueness we have

$$\Psi(x) = \Psi^\circ(x)/\Psi^\circ(+\infty),$$

where, from (3.6)

$$\begin{aligned}\Psi^\circ(+\infty) &:= \lim_{x \rightarrow +\infty} \Psi^\circ(x) \\ &= F(\alpha, \beta, \alpha + \beta + (2\mu/\theta); 1) \\ &= \frac{\Gamma(\alpha + \beta + (2\mu/\theta)) \Gamma(2\mu/\theta)}{\Gamma(\alpha + (2\mu/\theta)) \Gamma(\beta + (2\mu/\theta))},\end{aligned}$$

as claimed.  $\square$

**Example 5.2.** Choosing in (5.1)  $p = 0$ ,  $q = \gamma/a^2$ ,  $\theta = 1$ , and  $c = 1/a$  gives

$$\begin{aligned}\mathbf{E}_x \left( \exp \left( -\gamma \int_0^\infty (a + \exp(B_s^{(\mu)}))^{-2} ds \right) \right) \\ = K v(x)^{(\sqrt{a^2\mu^2 + 2\gamma} - a\mu)/a} F(\alpha, \beta, \alpha + \beta + 2\mu; v(x)),\end{aligned}$$

where

$$\alpha = \frac{1}{2} - \mu + \sqrt{\mu^2 + \frac{2\gamma}{a^2}} + \sqrt{\frac{1}{4} + \frac{2\gamma}{a^2}}, \quad \beta = \frac{1}{2} - \mu + \sqrt{\mu^2 + \frac{2\gamma}{a^2}} - \sqrt{\frac{1}{4} + \frac{2\gamma}{a^2}},$$

$K$  is given by (5.2) with  $\alpha$  and  $\beta$  as above, and

$$v(x) = \frac{\exp(x)}{a + \exp(x)}.$$

Notice that if  $\mu = 1/2$  then  $\beta = 0$  and because  $F(\alpha, 0, \gamma; x) = 1$  for all  $|x| < 1$  (see (3.2)) we obtain the result in [12] (see also [11]):

$$\mathbf{E}_x \left( \exp \left( -\gamma \int_0^\infty (a + \exp(B_s^{(1/2)}))^{-2} ds \right) \right) = v(x)^{(2a)^{-1}(\sqrt{a^2 + 8\gamma} - a)}.$$

From Proposition 2.3.2 it follows that

$$\int_0^\infty (a + \exp(B_s^{(\mu)}))^{-2} ds = H_0(Z) \quad \text{a.s.}$$

where  $Z$  satisfies the SDE

$$dZ_t = dB_t + \left( \left( \mu - \frac{1}{2} \right) \frac{a e^{aZ_t}}{1 - e^{aZ_t}} + a\mu \right) dt.$$

The function  $g$  is in this case

$$g(x) := -\frac{1}{a} \ln(1 + ae^{-x}).$$

**Example 5.3.** Let in (5.1)  $p + q = 0$ ,  $p = 4\rho \geq 0$ ,  $\theta = 2$ , and  $c = 1$ . Then

$$\alpha = \frac{1}{2} + \frac{1}{2}\sqrt{1-8\rho}, \quad \beta = \frac{1}{2} - \frac{1}{2}\sqrt{1-8\rho}, \quad k = 0,$$

$$K = \frac{\Gamma(\mu + \alpha) \Gamma(\mu + \beta)}{\Gamma(\mu) \Gamma(\mu + 1)},$$

and we obtain the result (see Vagurina [15])

$$\mathbf{E}_x \left( \exp \left( -\rho \int_0^\infty \cosh^{-2}(B_s^{(\mu)}) ds \right) \right) = K F(\alpha, \beta, 1 + \mu; v(x)).$$

By Proposition 2.3.2

$$\int_0^\infty \frac{ds}{\cosh^2(B_s^{(\mu)})} = H_\pi(Z), \quad a.s, \quad (5.11)$$

with  $Z$  determined via the SDE

$$dZ_t = dB_t + \left( \frac{1}{2} \operatorname{ctn} Z_t + \frac{\mu}{\sin Z_t} \right) dt, \quad Z_0 = 2\arctan \exp(B_0^{(\mu)}).$$

In this example

$$g(x) := 2\arctan e^x, \quad g'(x) = \frac{1}{\cosh x},$$

and, therefore,

$$G(g^{-1}(x)) = \frac{1}{2} \operatorname{ctn} x + \frac{\mu}{\sin x} = \frac{1}{2} \left( \mu - \frac{1}{2} \right) \tan \frac{x}{2} + \frac{1}{2} \left( \mu + \frac{1}{2} \right) \operatorname{ctn} \frac{x}{2}.$$

Further,

$$\mathbf{E}_z(\exp(-\rho H_\pi(Z))) = K F(\alpha, \beta, 1 + \mu; \sin^2(z/2))$$

**Example 5.4.** Take  $p = hc$  and  $q = a^2 c^2/2$  in the definition of  $I_\infty(p, q)$ . Then, letting  $c \rightarrow +\infty$ , we obtain by monotone convergence (using (1.1))

$$\begin{aligned} \lim_{c \rightarrow +\infty} \int_0^\infty \left( \frac{hc}{c \exp(\theta B_s^{(\mu)}) + 1} + \frac{a^2}{2} \frac{c^2}{(c \exp(\theta B_s^{(\mu)}) + 1)^2} \right) ds \\ = \int_0^\infty \left( h \exp(-\theta B_s^{(\mu)}) + \frac{a^2}{2} \exp(-2\theta B_s^{(\mu)}) \right) ds \end{aligned}$$

The Laplace transform of the functional on the right hand side is given in [3] formula 2.1.30.3(2) p. 292 and can be written as

$$\begin{aligned} \mathbf{E}_x \left( \exp \left( - \int_0^\infty \left( h \exp(-\theta B_s^{(\mu)}) + \frac{a^2}{2} \exp(-2\theta B_s^{(\mu)}) \right) ds \right) \right) \\ = \frac{\Gamma(1/2 + \mu/\theta + h/a\theta)}{\Gamma(2\mu/\theta)} \exp \left( - \frac{c}{\theta} e^{-\theta x} \right) \\ \times U \left( \frac{1}{2} - \frac{\mu}{\theta} + \frac{h}{a\theta}, 1 - \frac{2\mu}{\theta}, \frac{2a}{\theta} e^{-\theta x} \right), \end{aligned} \quad (5.12)$$

where the Kummer function  $U$  is connected to the Whittaker function  $W$  via

$$W_{n,m}(x) = W_{n,-m}(x) = x^{-m+1/2} e^{-x/2} U(-m-n+1/2, 1-2m, x),$$

see Abramowitz and Stegun [1]. To obtain the formula (5.12) from (5.1) observe first that as  $c \rightarrow +\infty$  we have  $\alpha \simeq 2ac/\theta$ ,

$$\beta \rightarrow \frac{1}{2} - \frac{\mu}{\theta} + \frac{h}{a\theta},$$

and

$$\left( 1 - \frac{1}{c e^{\theta x} + 1} \right)^{ac/\theta} \rightarrow \exp \left( - \frac{a}{\theta} e^{-\theta x} \right).$$

and recall that for  $\operatorname{Re} \gamma < 1$  and  $\operatorname{Re}(\alpha + \beta - \gamma) > -1$

$$\lim_{\alpha \rightarrow \infty} \frac{\Gamma(\alpha + 1 - \gamma)}{\Gamma(\alpha + \beta + 1 - \gamma)} F(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - \frac{x}{\alpha}) = U(\beta, \gamma, x),$$

see Abramowitz and Stegun [1].

## 6 Perpetual integral functional of BES(3)

Let  $R = \{R_t : t \geq 0\}$  denote a 3-dimensional Bessel process (or, equivalently, of index  $1/2$ ). The generator of  $R$  is

$$\mathcal{G}^R u = \frac{1}{2} \frac{d^2 u}{dx^2} + \frac{1}{x} \frac{du}{dx}.$$

In the next theorem we give the Laplace transform of the functional.

$$\widehat{I}_\infty(p, q) := \int_0^\infty \left( \frac{p}{\exp(\theta R_s) - 1} + \frac{q}{(\exp(\theta R_s) - 1)^2} \right) ds.$$

**Theorem 6.1.** *Let  $R$  be a 3-dimensional Bessel process started from  $x > 0$ . Then for  $\theta > 0$ ,  $p \geq 0$  and  $q \geq 0$*

$$\begin{aligned} \mathbf{E}_x \left( \exp \left( -\hat{I}_\infty(p, q) \right) \right) &= \frac{(1 - e^{-\theta x})^{\gamma/2} \Gamma(\alpha) \Gamma(\beta)}{x \theta \Gamma(\gamma)} F(\alpha, \beta, \gamma, 1 - e^{-\theta x}) \quad (6.1) \\ &= \frac{(1 - e^{-\theta x})^{\gamma/2}}{x \theta} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} (1-tx)^{\alpha-\gamma} dt, \end{aligned}$$

where

$$\alpha = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8q/\theta^2} + \frac{1}{\theta} \sqrt{2(q-p)} \quad (6.2)$$

$$\beta = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8q/\theta^2} - \frac{1}{\theta} \sqrt{2(q-p)}, \quad (6.3)$$

and  $\gamma = \alpha + \beta = 1 + \sqrt{1 + 8q/\theta^2}$ .

*Proof.* To start with, let  $\hat{Z}^\circ$  be a diffusion with the generator

$$\mathcal{G}^\circ u = \frac{1}{2} \frac{d^2 u}{dx^2} + h(x) \frac{du}{dx},$$

where  $\gamma > \alpha + \beta \geq 0$  and

$$h(x) := \frac{\theta}{2} \left( \frac{\gamma e^{\theta x}}{e^{\theta x} - 1} - (\alpha + \beta) \right).$$

Notice that if  $\gamma = \alpha + \beta$  then  $\hat{Z}^\circ$  coincides with the diffusion  $\hat{Z}$  introduced in Section 4.3. The measures induced by  $\hat{Z}^\circ$  and by  $R$  in the canonical space of continuous functions are absolutely continuous with respect to each other when restricted to the  $\sigma$ -algebra  $\mathcal{F}_t$  generated by the co-ordinate mappings up to a fixed but arbitrary time  $t$ . Let  $\mathbf{P}_x^{\hat{Z}}$  and  $\mathbf{P}_x^R$  be the measures associated with  $\hat{Z}^\circ$  and  $R$ , respectively, when both processes are started from  $x > 0$ . Then

$$\mathbf{P}_x^{\hat{Z}}|_{\mathcal{F}_t} = \exp(\hat{D}_t) \mathbf{P}_x^R|_{\mathcal{F}_t}, \quad (6.4)$$

where the exponent  $\hat{D}_t$  in the Radon–Nikodym derivative is given by

$$\hat{D}_t = - \int_0^t (\omega_s^{-1} - h(\omega_s)) d\omega_s + \frac{1}{2} \int_0^t (\omega_s^{-2} - h^2(\omega_s)) ds \quad (6.5)$$



Consider the stochastic integral term in (6.5). Under the measure  $\mathbf{P}_x^R$  the co-ordinate process is the Bessel process  $R$  started from  $x$ . Recalling that the quadratic variation of  $R$  is  $t$ , we obtain by Ito's formula

$$\begin{aligned} \text{(i)} \quad & -\int_0^t R_s^{-1} dR_s = \log\left(\frac{x}{R_t}\right) - \frac{1}{2} \int_0^t R_s^{-2} ds \\ \text{(ii)} \quad & \int_0^t h(R_s) dR_s = \frac{\gamma}{2} \log\left(\frac{e^{\theta R_t} - 1}{e^{\theta x} - 1}\right) - \frac{\theta(\alpha + \beta)}{2}(R_t - x) \\ & \quad + \frac{\theta^2 \gamma}{4} \int_0^t \frac{e^{\theta R_s}}{(e^{\theta R_s} - 1)^2} ds. \end{aligned}$$

Next consider the bounded variation part in (6.5). We have

$$\begin{aligned} \text{(iii)} \quad & -\frac{1}{2} \int_0^t h^2(R_s) ds = -\frac{\theta^2 \gamma^2}{8} \int_0^t \frac{e^{2\theta R_s}}{(e^{\theta R_s} - 1)^2} ds - \frac{\theta^2(\alpha + \beta)^2}{8} t \\ & \quad + \frac{\theta^2 \gamma(\alpha + \beta)}{4} \int_0^t \frac{e^{\theta R_s}}{e^{\theta R_s} - 1} ds. \end{aligned}$$

Because  $R_t \rightarrow +\infty$  as  $t \rightarrow +\infty$ , it is seen from (ii) and (iii) above that taking  $\gamma = \alpha + \beta$  will lead to a remarkably simple special case. Indeed, if  $\gamma = \alpha + \beta$

$$\begin{aligned} \widehat{D}_t &= \log\left(\frac{x}{R_t}\right) + \frac{\gamma}{2} \log\left(\frac{1 - e^{-\theta R_t}}{1 - e^{-\theta x}}\right) + \frac{\theta^2 \gamma}{8} \int_0^t (e^{\theta R_s} - 1)^{-1} ds \\ & \quad - \frac{\theta^2 \gamma}{4} \left(\frac{\gamma}{2} - 1\right) \int_0^t (e^{\theta R_s} - 1)^{-2} ds \end{aligned}$$

In view of Proposition 4.5 we consider now the functional (assume that the continuous function  $\omega$  is such that the functional is well-defined)

$$\widehat{I}_t := \frac{\theta^2}{2} \int_0^t (\exp(\theta \omega_s) - 1)^{-1} ds.$$

By absolute continuity,

$$\mathbf{E}_x^{\widehat{Z}} \left( \exp(-r \widehat{I}_t) \right) = \mathbf{E}_x^R \left( \exp(\widehat{D}_t - r \widehat{I}_t) \right),$$

and, further, for  $H_y := \inf\{t : \omega_t = y\}$  with  $y > x$

$$\mathbf{E}_x^{\hat{Z}} \left( \exp(-r \hat{I}_{H_y}) \right) = \mathbf{E}_x^R \left( \exp(\hat{D}_{H_y} - r \hat{I}_{H_y}) \right).$$

Proposition 4.5 gives now

$$\mathbf{E}_{x^*}^X (\exp(-r H_{y^*})) = \mathbf{E}_x^R \left( \exp(\hat{D}_{H_y} - r \hat{I}_{H_y}) \right), \quad (6.6)$$

where  $x^* = 1 - e^{-\theta x}$ ,  $y^* = 1 - e^{-\theta y}$ , and  $X$  is a Jacobi diffusion with parameters  $\alpha, \beta$ , and  $\gamma = \alpha + \beta \geq 2$ . The identity (6.6) is equivalent with

$$\mathbf{E}_x^R \left( \exp(-\hat{I}_{H_y}(p, q)) \right) = \frac{y}{x} \left( \frac{1 - e^{-\theta x}}{1 - e^{-\theta y}} \right)^{\gamma/2} \mathbf{E}_{x^*}^X (\exp(-r H_{y^*})), \quad (6.7)$$

where

$$p = \frac{\theta^2}{4} (2r - \gamma) \quad \text{and} \quad q = \frac{\theta^2}{8} \gamma (\gamma - 2). \quad (6.8)$$

Letting  $r = \alpha\beta$  it is seen from (6.8), after straightforward computations, that  $\alpha$  and  $\beta$  can be expressed as in (6.2) and (6.3), respectively. To conclude the proof it remains to compute the Laplace transform on the right hand side of (6.7) and let  $y \rightarrow +\infty$ . Because  $x < y$  we have also  $x^* < y^*$ , and, hence,

$$\mathbf{E}_{x^*}^X (\exp(-\alpha\beta H_{y^*})) = \frac{\psi_{\alpha\beta}^X(x^*)}{\psi_{\alpha\beta}^X(y^*)}, \quad (6.9)$$

where  $\psi_{\alpha\beta}^X$  is the unique (up to a multiple), positive increasing solution of (3.2) with  $\gamma = \alpha + \beta$ . We remark that the uniqueness follows from the fact that both boundaries, in this case, are entrance-not-exit. Moreover, because  $H_{y^*} \rightarrow +\infty$  as  $y^* \rightarrow 1$  (also as  $y^* \rightarrow 0$ ) it must hold that  $\psi_{\alpha\beta}^X(y^*) \rightarrow +\infty$  as  $y^* \rightarrow 1$ . Similarly, there exists a unique positive decreasing solution  $\varphi_{\alpha\beta}^X$  such that  $\varphi_{\alpha\beta}^X(y^*) \rightarrow +\infty$  as  $y^* \rightarrow 0$ . From the general theory of differential equations it follows that all other solutions of (3.2) can be expressed as linear combinations of  $\psi_{\alpha\beta}^X$  and  $\varphi_{\alpha\beta}^X$ . Recall (see Section 3) that the function  $x \mapsto F(\alpha, \beta, \gamma; x)$  is a solution of (3.2). Now  $\gamma = \alpha + \beta$  and from Abramowitz and Stegun [1] 15.3.10 p. 559

$$F(\alpha, \beta, \alpha + \beta; x) \simeq -\frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \log(1 - x), \quad \text{as } x \uparrow 1, \quad (6.10)$$

and, because  $F(\alpha, \beta, \alpha + \beta; 0) = 1$ , it follows that  $x \mapsto F(\alpha, \beta, \alpha + \beta; x)$  is increasing, and, consequently,

$$\psi_{\alpha\beta}^X = F(\alpha, \beta, \alpha + \beta; \cdot).$$

Therefore, from (6.9) and (6.7),

$$\mathbf{E}_x^R \left( \exp(-\widehat{I}_{H_y}(p, q)) \right) = \frac{y}{x} \left( \frac{1 - e^{-\theta x}}{1 - e^{-\theta y}} \right)^{\gamma/2} \frac{F(\alpha, \beta, \alpha + \beta; x^*)}{F(\alpha, \beta, \alpha + \beta; y^*)}.$$

Letting here  $y \rightarrow +\infty$  and using (6.10) proves the claim.  $\square$

**Remark 6.2.** The decreasing solution  $\varphi_{\alpha\beta}^X$  is given by

$$\varphi_{\alpha\beta}^X(x) = \widehat{F}(\alpha, \beta, \alpha + \beta; x) := F(\alpha, \beta, 1; 1 - x).$$

From the definition of  $F$  and the facts that, in this case,  $\alpha\beta > 0$  and

$$(\alpha + k)(\beta + k) = \alpha\beta + k(\alpha + \beta) + k^2,$$

it is seen that  $\widehat{F}$  has the desired properties, i.e.,

$$\lim_{x \rightarrow 0} \widehat{F}(\alpha, \beta, \alpha + \beta; x) = +\infty \quad \text{and} \quad \widehat{F}(\alpha, \beta, \alpha + \beta; 1) = 1,$$

implying that  $\widehat{F}$  is decreasing.

**Example 6.3.** In the particular case  $p = q = 4\rho$ , and  $\theta = 2$ , we have

$$\alpha = \beta = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 8\rho}$$

and, therefore

$$\mathbf{E}_x \exp \left( - \int_0^\infty \frac{\rho ds}{\sinh^2(R_s)} \right) = \frac{(1 - e^{-2x})^{2\alpha} \Gamma^2(\alpha)}{2x \Gamma(2\alpha)} F(\alpha, \alpha, 2\alpha; 1 - e^{-2x})$$

**Example 6.4.** Choosing  $q = 0$  yields

$$\alpha = 1 + \frac{i}{\theta} \sqrt{2p}, \quad \beta = 1 - \frac{i}{\theta} \sqrt{2p} =: \bar{\alpha}.$$

Hence,  $\alpha + \beta = 2$  and

$$\begin{aligned} \mathbf{E}_x \exp\left(-\int_0^\infty \frac{p}{\exp(\theta R_s) - 1} ds\right) \\ = \frac{1 - e^{-\theta x}}{x \theta} \Gamma(\alpha) \Gamma(\bar{\alpha}) F(\alpha, \bar{\alpha}, 2; 1 - e^{-\theta x}) \\ = \frac{1 - e^{-\theta x}}{x \theta} \frac{\pi \sqrt{2p}}{\theta \sinh(\pi \sqrt{2p}/\theta)} F(\alpha, \bar{\alpha}, 2; 1 - e^{-\theta x}), \end{aligned}$$

where the formula (see [1] formulae 6.1.28 and 6.1.29.)

$$\Gamma(1 + iy) \Gamma(1 - iy) = \frac{\pi y}{\sinh(\pi y)}$$

is used. Letting  $x \rightarrow 0$  yields

$$\mathbf{E}_0 \exp\left(-\int_0^\infty \frac{p}{\exp(\theta R_s) - 1} ds\right) = \frac{\pi \sqrt{2p}}{\theta \sinh(\pi \sqrt{2p}/\theta)}. \quad (6.11)$$

The term on the right hand side is the Laplace transform of  $H_{\pi/\theta}(R)$  where the BES(3) process  $R$  is started at 0, i.e., (6.11) is equivalent with the following identity due to Donati–Martin and Yor [4] p. 1044:

$$\int_0^\infty \frac{1}{\exp(\theta R_s) - 1} ds \stackrel{(d)}{=} H_{\pi/\theta}(R). \quad (6.12)$$

The derivation of (6.12) in [4] is very different than the one presented above, and is based on a formula for the Laplace transform of an integral functional of a two-dimensional Bessel process.

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